

Log-periodic oscillations of transverse momentum distributions

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Abstract

Large p_T transverse momentum distributions apparently exhibit power-like behavior. We argue that, under closer inspection, this behavior is in fact decorated with some log-periodic oscillations. Assuming that this is a genuine effect and not an experimental artefact, it suggests that either the exponent of the power-like behavior is in reality complex, or that there is a scale parameter which exhibits specific log-periodic oscillations. This problem is discussed using Tsallis distribution with scale parameter T . At this stage we consider both possibilities on equal footing.

Keywords: scale invariance, log-periodic oscillation, $p - p$ collisions

For some time now it has been popular to fit the different kinds of transverse momentum spectra measured in multiparticle production processes to of a Tsallis formula [1] (cf., for example, [2, 3, 4, 5, 6, 7, 8]). It can be written in one of two recognized forms: either in original Tsallis one (with two parameters: q and T),

$$f(p_T) = C \cdot \left[1 - (1 - q) \frac{p_T}{T} \right]^{\frac{1}{1-q}} \quad (1)$$

or, in the so called "QCD inspired" Hagedorn form [9, 10] (with parameters: m and T):

$$h(p_T) = C \left(1 + \frac{p_T}{mT} \right)^{-m}; \quad m = \frac{1}{q-1}. \quad (2)$$

For our purposes, these are equivalent (and we shall use them interchangeably.) They both represent the simplest way of describing the whole observed range of measured p_T distributions. The best examples are the recent successful fits [11] to very large p_T data measured by the LHC experiments CMS [12], ATLAS [13] and ALICE [14] for pp collisions, see Fig. 1¹.

Albeit both fits look pretty good, closer inspection shows that ratio of data/fit is not flat but shows some kind of clearly visible oscillations, cf. Fig. 2. It turns out that they cannot be eliminated by suitable changes of parameters (q, T) or (m, T) in Eqs. (1) or (2), respectively². In what follows, we assume that this observation is not an experimental artifact but rather it represents some genuine dynamical effect which is worth investigating in more detail.

First notice, that to account for these fluctuations of $f(p_T)$ from Eq. (1) (or $h(p_T)$ from Eq. (2)) the original Tsallis formula has to be multiplied by the following factor (log-oscillating function):

$$R(E) = a + b \cos [c \ln(E + d) + f] \quad (3)$$

approach. Nevertheless, it turns out that even from this kind of approach one can get a distribution with essentially only two parameters (not counting normalization) which closely resembles the usual Tsallis distribution (1), albeit it is not identical with it [15]. Notice that at the midrapidity, i.e., for $y \simeq 0$, and for large transverse momenta, $p_T > M$, one has $E = \sqrt{M^2 + p_T^2} \cosh(y) \simeq p_T$.

²One has to realize that to really see these oscillations one needs rather large domain in p_T . Therefore, albeit similar effects can be also seen at lower energies, they are not so pronounced as here and therefore will not be discussed at this point.

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¹This is the usual domain reserved for the purely perturbative QCD

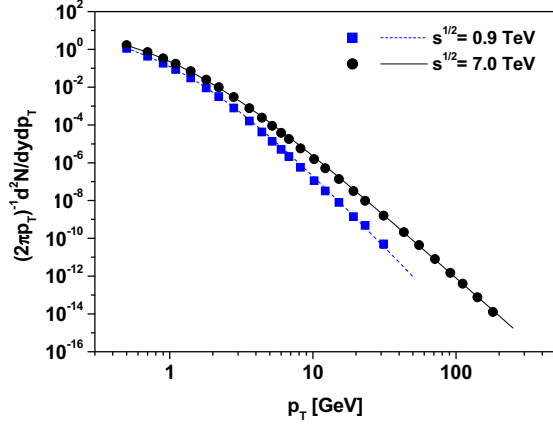


Figure 1: (Color online) Fit to large p_T data for pp collisions at 0.9 and 7 TeV from CMS experiment using distribution (2) [12]. Parameters used are, respectively, ($T = 0.135$, $m = 8$) and ($T = 0.145$, $m = 6.7$).

As recently shown in [16], such a factor, dressing the original power law distribution (in our case quasi-power law Tsallis distribution (1)), arises in a natural way if one allows the power index q to be complex³. For completeness, we shortly explain what this means. In general, if some function $O(x)$ is scale invariant, i.e., if $O(x) = \mu O(\lambda x)$, then it must have a power law behavior,

$$O(x) = Cx^{-m} \quad \text{with} \quad m = \frac{\ln \mu}{\ln \lambda}. \quad (4)$$

Because one can write $\mu\lambda^{-m} = 1 = e^{i2\pi k}$, where k is an arbitrary integer, in general,

$$m = -\frac{\ln \mu}{\ln \lambda} + i\frac{2\pi k}{\ln \lambda}. \quad (5)$$

As shown in [16], the evolution of the differential $df(E)/dE$ of a Tsallis distribution $f(E)$ with power index n performed for finite differences $dE = \alpha(nT + E)$ (where $\alpha < nT$ is another new parameter) results in the following scale invariant relation

$$g[(1 + \alpha)x] = (1 - \alpha n)g(x) \quad (6)$$

where $x = 1 + E/(nT)$. This means that, in general, one

³There is vast literature of such situation in different branches of physics, cf. [17] and other references [16].

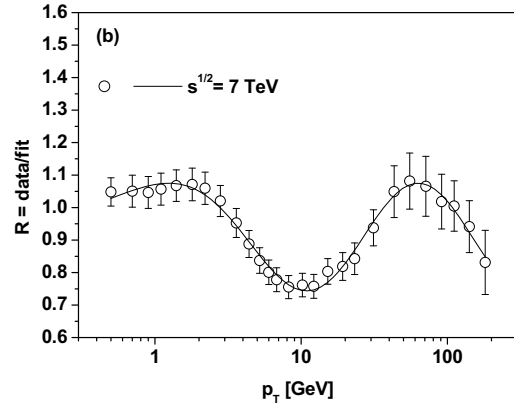
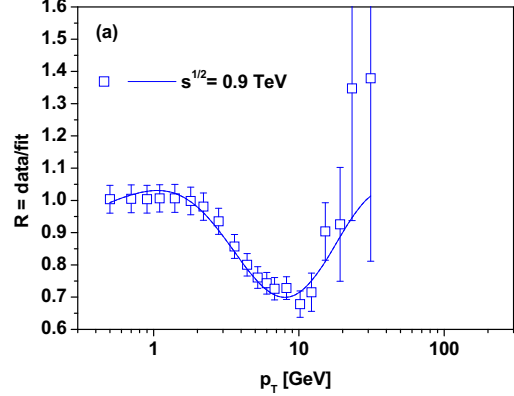


Figure 2: (Color online) Fit to p_T dependence of data/fit ratio for results from Fig. 1. Parameters of function R used (3) here are, respectively: $a = 0.865$, $c = 2.1$ for 0.9 TeV and $a = 0.909$, $c = 1.86$ for 7 TeV, whereas for both energies $b = 0.166$, $d = 0.948$ and $f = -1.462$.

can write Eq. (1) in the form:

$$g(x) = x^{-m_k}, \quad m_k = -\frac{\ln(1 - \alpha n)}{\ln(1 + \alpha)} + ik\frac{2\pi}{\ln(1 + \alpha)}. \quad (7)$$

The power index in Eq. (7) (and in Eq. (1)) is therefore a complex number, the imaginary part of which signals a hierarchy of scales leading to the log-periodic oscillations. The meaning of the parameter α becomes clear by noticing that in the special case of $k = 0$, for which one recovers the usual real power law solution, m_0 corresponds

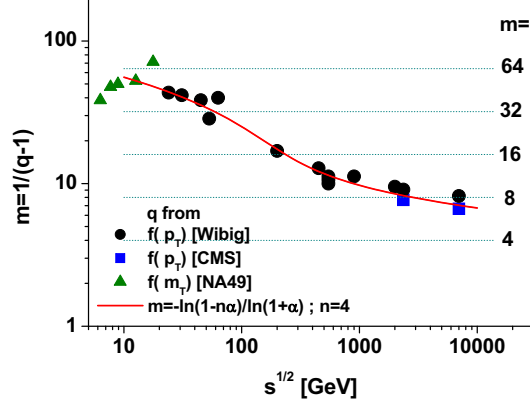


Figure 3: (Color online) The energy dependence of $m = m_0$ deduced from data [20, 12, 21].

to fully continuous scale invariance⁴. In this case one recovers in the limit $\alpha \rightarrow 0$ the power n in the usual Tsallis distribution. However, in general one has

$$\begin{aligned} g(x) &= \sum_{k=0} w_k \cdot \text{Re}(x^{-m_k}) = \\ &= x^{-\text{Re}(m_k)} \sum_{k=0} w_k \cdot \cos[\text{Im}(m_k) \ln(x)]. \end{aligned} \quad (8)$$

This is a general form of a Tsallis distribution for complex values of the nonextensivity parameter q . It consists of the usual Tsallis form (albeit with a modified power exponent) and a dressing factor which has the form of a sum of log-oscillating components, numbered by parameter k . Because we do not know *a priori* the details of dynamics of processes under consideration (i.e., we do not know the weights w_k), in what follows we use only $k = 0$ and $k = 1$ terms. We obtain approximately,

$$g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos \left[\frac{2\pi}{\ln(1+\alpha)} \ln \left(1 + \frac{E}{nT} \right) \right] \right\}. \quad (9)$$

In this case one could expect that parameters in general modulating factor R in Eq. (3) could be identified as fol-

⁴In this case power law exponent m_0 still depends on α and increases with it roughly as $m_0 \simeq n + \frac{n}{2}(n+1)\alpha + \frac{n}{12}(4n^2 + 3n - 1)\alpha^2 + \frac{n}{24}(6n^3 + 4n^2 - n + 1)\alpha^3 + \dots$. Notice also that $\alpha < 1/n$.

lows:

$$a = w_0, \quad b = w_1, \quad c = \frac{2\pi}{\ln(1+\alpha)}, \quad d = nT, \quad f = -c \cdot \ln(nT). \quad (10)$$

Comparison of the fit parameters of the oscillating term R in Eq. (3) with Eq. (7) clearly shows that the observed frequency, here given by the parameter c , is more than an order of magnitude smaller than the expected value equal to $2\pi / \ln(1+\alpha)$ for any reasonable value of α . To explain this, notice that in our formalism leading to Eq. (9) only one evolution step is assumed, whereas in reality we have a whole hierarchy of κ evolutions. This results (cf. [16]) in the scale parameter c being κ times smaller than in (9),

$$c = \frac{2\pi}{\kappa \ln(1+\alpha)}. \quad (11)$$

Experimental data indicate that $\kappa \simeq 22$ (for $\alpha \simeq 0.15$ and $c \simeq 2$).

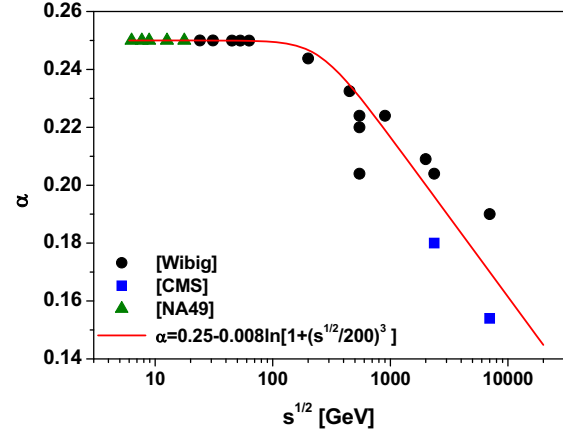


Figure 4: (Color online) The energy dependence of parameter α present in m_0 plotted in Fig. 3.

From Eq.(7) we see that $m_0 > n$. This suggests the following explanation of the difference seen between prediction from theory and the experimental data: the measurements in which log-periodic oscillations appear underestimate the true value that follows from the underlying dynamics which leads to the smooth Tsallis distribution. As an example consider the m_0 dependence on α , assuming the initial slope $n = 4$ (this is the value of n expected from the pure QCD considerations for partonic interactions [15]). The energy behavior of the power index m_0

in the Tsallis part is shown in Fig. 3, whereas the energy dependence of the parameter α contained in m_0 is shown in Fig. 4.

So far we attributed the observed log-periodic oscillations to the complex values of the power index m (i.e., to the complex nonextensivity parameter q)⁵. However, this phenomenon can be also explained in a completely different way, namely by keeping the nonextensivity parameter q real (as in the original Tsallis distribution) but instead allowing the parameter T to oscillate in a specific way as displayed in Fig. 5. As seen there, the observed log-periodic oscillations of R can be reproduced by a suitable p_T dependence of the scale parameter (the temperature) T , present in Tsallis distribution, here expressed by following a general formula (resembling Eq. (3), with generally energy dependent fit parameters $(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{f})$):

$$T = \bar{a} + \bar{b} \sin \left[\bar{c} (\ln(E + \bar{d})) + \bar{f} \right] \quad (12)$$

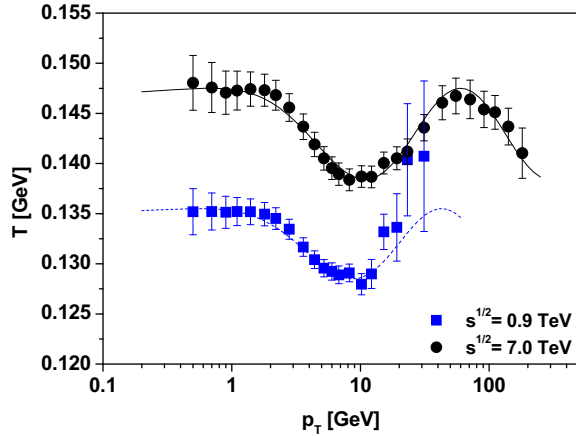


Figure 5: (Color online) The $T = T(p_T)$ for Eq. (12) for which $R = 1$. Parameters used are: $\bar{a} = 0.132$, $\bar{b} = 0.0035$, $\bar{c} = 2.2$, $\bar{d} = 2.0$, $\bar{f} = -0.5$ for 0.9 TeV and $\bar{a} = 0.143$, $\bar{b} = 0.0045$, $\bar{c} = 2.0$, $\bar{d} = 2.0$, $\bar{f} = -0.4$ for 7 TeV.

To explain such behavior, start with the well known [18] stochastic equation for the temperature evolution, which in the Langevin formulation has the form (in which

we allow for an energy dependent noise, $\xi(t, E)$):

$$\frac{dT}{dt} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (13)$$

For the time dependent $E = E(t)$ it reads:

$$\frac{dT}{dE} \frac{dE}{dt} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (14)$$

In the scenario of *preferential attachment* (known from the growth of networks [19]) one expects that⁶

$$\frac{dE}{dt} = \frac{E}{n} + T. \quad (15)$$

Using now Eq. (15) one can write Eq. (14 as

$$\left(\frac{E}{n} + T \right) \frac{dT}{dE} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (16)$$

This can be subsequently transformed to

$$\left(\frac{1}{n} + T e^{-\ln E} \right) \frac{dT}{d(\ln E)} + \frac{1}{\tau} T + \xi(t, E) T = \Phi \quad (17)$$

and, after differentiating, to

$$\begin{aligned} & \left(\frac{1}{n} + T e^{-\ln E} \right) \frac{d^2 T}{d(\ln E)^2} + \left[\frac{dT}{d(\ln E)} \right]^2 e^{-\ln E} - \\ & - \left[T e^{-\ln E} - \frac{1}{\tau} - \xi(t, E) \right] \frac{dT}{d(\ln E)} + \\ & + T \frac{d\xi(t, E)}{d(\ln E)} = 0. \end{aligned} \quad (18)$$

For large E (i.e., neglecting terms $\propto 1/E$) one obtains the following equation for T :

$$\frac{1}{n} \frac{d^2 T}{d(\ln E)^2} + \left[\frac{1}{\tau} + \xi(t, E) \right] \frac{dT}{d(\ln E)} + T \frac{d\xi(t, E)}{d \ln E} = 0. \quad (19)$$

Now assume that noise $\xi(t, E)$ increases logarithmically with energy,

$$\xi(t, E) = \xi_0(t) + \frac{\omega^2}{n} \ln E. \quad (20)$$

⁵The possible dynamical implication of this fact, cf. for example [17, 16] and remarks in footnote 7, is outside of the scope of present paper.

⁶Notice that in the usually used multiplicative noise scenario described by $\gamma(t)$, not discussed here, one has $\frac{dE}{dt} = \gamma(t)E + \xi(t)$.

For this choice of noise Eq. (19) is just an equation for the damped hadronic oscillator and has a solution in the form of log-periodic oscillation of temperature with frequency ω :

$$T = C \exp \left\{ -n \cdot \left[\frac{1}{2\tau} + \frac{\xi(t, E)}{2} \right] \ln E \right\} \cdot \sin(\omega \ln E + \phi). \quad (21)$$

The phase shift parameter ϕ depends on the unknown initial conditions and is therefore an additional fitting parameter. Averaging the noise fluctuations over time t and taking into account that the noise term cannot on average change the temperature (cf. Eq. (13) in which $\langle dT/dt \rangle = 0$ for $\Phi = 0$), i.e., that

$$\frac{1}{\tau} + \langle \xi(t, E) \rangle = 0, \quad (22)$$

we have

$$T = \bar{a} + \frac{b'}{n} \sin(\omega \ln E + \phi). \quad (23)$$

The amplitude of oscillations, b'/n , comes from the assumed behavior of the noise as given in Eq. (20). Notice that for large n , the energy dependence of the noise disappears (and because, in general, n decreases with energy, one can therefore expect only negligible oscillations for lower energies but increasing with the energy). This should now be compared with the parametrization of $T(E)$ given by Eq. (12) and used to fit data in Fig.5. Looking at parameters we can see that only a small amount of T (of the order of $\bar{b}/\bar{a} \sim 3\%$) comes from the stochastic process with energy dependent noise, whereas the main contribution emerges from the usual energy-independent Gaussian white noise.

To conclude, the above oscillating T needed to fit the log-periodic oscillations seen in data can be obtained in yet another way. So far we were assuming that the noise $\xi(t, E)$ has the form of Eq. (20) and, at the same time, we were keeping the relaxation time τ constant. However, it turns out that we could equivalently assume the energy E independent white noise, $\xi(t, E) = \xi_0(t)$, but allow for the energy dependent relaxation time taken in the form of

$$\tau = \tau(E) = \frac{n\tau_0}{n + \omega^2 \ln E}. \quad (24)$$

This assumption corresponds to the following time evolution of the temperature,

$$T(t) = \langle T \rangle + [T(t=0) - \langle T \rangle] E^{-t\omega^2/n} \exp\left(-\frac{t}{\tau_0}\right), \quad (25)$$

which is gradually approaching its equilibrium value $\langle T \rangle$ and reaches it more quickly for higher energies.

To summarize, we have presented two possible mechanisms which could result in the log-periodic oscillations apparently present in data for transverse momentum distributions observed in LHC experiments. In both cases one uses a Tsallis formula (either in the form of Eq. (1) or Eq. (2)), with main parameters m - the scaling power exponent (or nonextensivity $q = 1 + 1/m$) and T - the scale parameter (temperature). In the first approach, our Tsallis distribution is decorated by the oscillating factor which emerges in a natural way in the case of complex power exponent m (or complex nonextensivity q)⁷ with the scale parameter T remaining untouched. In the second approach, it is the other way around, i.e., whereas m (or n as in Eq. (21)) remains untouched, the scale parameter T is now oscillating. From Eq. (23) one can see that $T = T(n = 1/(1 - q), E)$ and as a function of nonextensivity parameter q it continues our previous efforts to introduce an effective temperature into the Tsallis distribution, $T_{eff} = T(q)$, here in a much more general form as in [2] or [25]. The two possible mechanisms resulting in such T were outlined: the energy dependent noise connected with the constant relaxation time, or else the energy independent white noise, but with energy dependent relaxation time.

We close by noting that, at the present level of investigation, we are not able to indicate which of the two possible mechanisms presented here (complex q or oscillating T) and resulting in log-periodic oscillations is the preferred one. This would demand more detailed studies on the possible connections with dynamical pictures. For example, as discussed long time ago by studying apparently similar effects in some exclusive reactions using the QCD Coulomb phase shift idea [26]. The occurrence of some kind of complex power exponents was noticed there as well, albeit on completely different grounds than in our case. A possible link with our present analysis would be very interesting but would demand an involved and thor-

⁷It is worth to mention at this point that complex q inevitably means also complex heat capacity $C = 1/(1 - q)$ (c.f., [2, 22] and also [23]). Such complex (frequency dependent) heat capacities are widely known and investigated, see [24].

ough analysis.

Acknowledgments

This research was supported in part by the National Science Center (NCN) under contract Nr 2013/08/M/ST2/00598. We would like to warmly thank Dr Eryk Infeld for reading this manuscript.

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